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RIEMANNIAN MANIFOLDS ADMITTING AN INFINITESIMAL CONFORMAL TRANSFORMATION

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1. Introduction

Let M be an n-dimensional connected Riemannian manifold with positive definite metric of differentiability class C^∞. We cover M by a system of coordinate neighborhoods $\{U; x^h\}$, and denote by $g_{ji}, \nabla_i, K_{kji}{}^h, K_{ji}$ and K the fundamental metric tensor field, the operator of covariant differentiation with respect to the Levi-Civita connection, the curvature tensor field, the Ricci tensor field and the scalar curvature field of M respectively. Here and in the sequel indices $h, i, j, k, \cdots$ run over the range $\{1, \cdots, n\}$.

We denote by $C_0(M)$ the largest connected group of conformal transformations of a Riemannian manifold M, and by $I_0(M)$ the largest connected group of isometries of M.

Riemannian manifolds with constant scalar curvature field admitting an infinitesimal nonhomothetic conformal transformation have been extensively studied and we know the following theorems.

Theorem A (*Yano and Nagano* [38]). *If M is a complete Einstein manifold of dimension $n > 2$ and*

$$C_0(M) \neq I_0(M) , \tag{1.1}$$

then M is isometric to a sphere.

(See also Bishop and Goldberg [3].)

Theorem B (*Nagano* [23]). *If M is a complete Riemannian manifold of dimension $n > 2$ with parallel Ricci tensor field and* (1.1) *holds, then M is isometric to a sphere.*

Theorem C (*Goldberg and Kobayashi* [5], [6], [7]). *If M is a compact homogeneous Riemannian manifold of dimension $n > 3$, and* (1.1) *holds, then M is isometric to a sphere.*

Theorem D (*Lichnerowicz* [22]). *If M is a compact Riemannian manifold of dimension $n > 2$, $K =$ const., and $K_{ji}K^{ji} =$ const., then* (1.1) *implies that M is isometric to a sphere.*

Theorem E (*Hsiung* [11], [12], [13]). *If M is compact and of dimension*

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$n > 2$, $K = const.$, and $K_{kjih}K^{kjih} = const.$, then (1.1) *implies that M is isometric to a sphere.*

Theorem F (*Obata* [27], *Yano* [33]). *If M is compact, orientable and of dimension $n > 2$ with constant K, and admits an infinitesimal nonhomothetic conformal transformation v^h so that*

$$\mathscr{L}_v g_{ji} = 2\rho g_{ji} , \tag{1.2}$$

$\mathscr{L}_v$ denoting the Lie derivation with respect to v^h, such that

$$\int_M G_{ji}\rho^j\rho^i dv \geq 0 , \tag{1.3}$$

where

$$G_{ji} = K_{ji} - \frac{1}{n}Kg_{ji} , \tag{1.4}$$

and $\rho^j = g^{ji}\rho_i$, $\rho_i = \nabla_j\rho$, dV being the volume element of M, then M is isometric to a sphere.

Theorem G (*Yano* [33]). *If M is compact and of dimension $n > 2$ with constant K, and admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2) *such that*

$$\mathscr{L}_v(G_{ji}G^{ji}) = 0 \tag{1.5}$$

or

$$\mathscr{L}_v(Z_{kjih}Z^{kjih}) = 0 , \tag{1.6}$$

where

$$Z_{kji}{}^h = K_{kji}{}^h - \frac{K}{n(n-1)}(\delta_k^h g_{ji} - \delta_j^h g_{ki}) , \tag{1.7}$$

then M is isometric to a sphere.

(See also Hiramatu [10].)

Theorem G, which is a generalization of Theorem D and Theorem E, has been further generalized by Obata and one of the present authors [40].

Theorem H (*Goldberg* [4]). *If M is compact and of dimension $n > 2$ with constant K, and admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2), *then*

$$K^2\rho^2 \leq n(n-1)^2(\nabla_j\rho_i)(\nabla^j\rho^i) , \tag{1.8}$$

where $\nabla^j = g^{ji}\nabla_i$, equality holding if and only if M is isometric to a sphere.

One of the present authors showed that the compactness here can be replaced by completeness (Yano [34]).

Theorem I (*Yano* [34]). *If M is compact, orientable and of dimension $n > 2$ with constant K, and admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2), *then*

$$n(n-1)\int_M K_{ji}\rho^j\rho^i dV \leq K^2 \int_M \rho^2 dV \,, \tag{1.9}$$

equality holding if and only if M is isometric to a sphere.

(See also Hiramatu [9].)

The assumption $K =$ const. in all the above theorems is based on the following result of Yamabe [30].

Theorem J. *For any Riemannian metric given on a compact C^∞-differentiable manifold of dimension $n \geq 3$, there always exists a Riemannian metric which is conformal to the given metric and whose scalar curvature field is a constant.*

To prove that a complete Riemannian manifold is isometric to a sphere, the following theorem due to Obata [24], [25], [26] is very useful:

Theorem K. *If a complete Riemannian manifold M of dimension $n \geq 2$ admits a nonconstant function ρ such that*

$$\nabla_j \nabla_i \rho = -c^2 \rho g_{ji} \,, \tag{1.10}$$

where c is a positive constant, then M is isometric to a sphere of radius $1/c$ in $(n+1)$-dimensional Euclidean space.

One of the present authors tried to replace the condition $K =$ const. in above theorems by

$$\mathscr{L}_v K = 0 \,, \tag{1.11}$$

and obtained the following theorems.

Theorem L (*Yano* [35]). *If M is a compact orientable Riemannian manifold of dimension $n > 2$, and admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2), (1.11) *and*

$$\int_M \left(K_{ji}\rho^j\rho^i - \frac{1}{n(n-1)} K^2\rho^2\right) V \geq 0 \,, \tag{1.12}$$

then M is conformal to a sphere.

Theorem M (*Yano* [35]). *If M is a compact orientable Riemannian manifold and of dimension $n > 2$, and admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2) *such that* (1.11), (1.5) *and*

(1.13) $$\frac{1}{n-1}\int_M K^2\rho^2 dV \leq \int_M K\rho_i\rho^i dV ,$$

or (1.11), (1.6) *and* (1.13) *hold, then M is conformal to a sphere.*

We note here that the conditions (1.11), (1.5) and (1.11), (1.6) are respectively equivalent to the conditions

$$\mathscr{L}_v K = 0 , \quad \mathscr{L}_v(K_{ji}K^{ji}) = 0 \quad \text{and} \quad \mathscr{L}_v K = 0 , \quad \mathscr{L}_v(K_{kjih}K^{kjih}) = 0 .$$

To prove these theorems, the following theorem due to Tashiro (see [29] and also Ishihara [18], Ishihara and Tashiro [19]) is used.

Theorem N. *If a compact Riemannian manifold M of dimension $n \geq 2$ admits a nonconstant function ρ such that*

(1.14) $$\nabla_j \nabla_i \rho = \frac{1}{n}\Delta\rho g_{ji} ,$$

then M is conformal to a sphere in $(n+1)$-dimensional Euclidean space.

Sawaki and one of the present authors [42] proved the following three theorems.

Theorem O. *If a complete Riemannian manifold M of dimension $n \geq 2$ admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2) *and* (1.1), *then we have* (1.8) *where the equality holds if and only if M is isometric to a sphere.*

Theorem P. *If a compact Riemannian manifold M of dimension $n \geq 2$ admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2), (1.11) *and*

(1.15) $$K_i{}^h\rho^i = k\rho^h ,$$

k being a constant satisfying

(1.16) $$K^2 \leq n^2k^2 ,$$

then M is isometric to a sphere.

Theoerm Q. *If a compact orientable Riemannian manifold M of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2) *and* (1.11), *then*

(1.17) $$n(n-1)\int_M K_{ji}\rho^j\rho^i dV \leq \int_M K^2\rho^2 dV ,$$

equality holding if and only if M is isometric to a sphere.

Hsiung and Stern [16], [17] proved

Theorem R. *Suppose that a compact Riemannian manifold M of dimen-*

sion $n > 2$ admits an infinitesimal nonhomothetic conformal transformation v^h satisfying (1.2) *and* (1.11). *If one of the following conditions is satisfied, then M is conformal to a sphere*:

$$\nabla_j \nabla_i F = K\rho g_{ji}, \ F \text{ being a scalar field on } M\ , \tag{1.18}$$

$$K_{ji}\rho^i = \frac{1}{n}\nabla_j(K\rho) \quad \text{and} \quad \nabla_j\nabla_i(K\rho) = K\nabla_j\nabla_i\rho\ , \tag{1.19}$$

$$\mathscr{L}_v K_{ji} = \alpha g_{ji}, \ \alpha \text{ being a scalar field on } M\ . \tag{1.20}$$

For generalizations of the above theorems to the case of conformal changes of metric, see Barbance [2], Goldberg and Yano [8], Hsiung and Liu [14], Hsiung and Mugridge [15] and Yano and Obata [40], and for further results on conformal transformations see Yano [36], [37].

The purpose of the present paper is to eliminate the condition $K =$ const. or $\mathscr{L}_v K = 0$ in the above theorems concerning Riemannian manifolds admitting an infinitesimal conformal transformation.

In the sequel, we need the following theorem due to Tashiro [29]:

Theorem S. *If a complete Riemannian manifold M of dimension $n > 2$ admits a complete vector field v^h satisfying* (1.2) *and* (1.14) *with nonconstant ρ, then M is isometric to a sphere.*

2. Lemmas

Lemma 1 (*Lichnerowicz* [21], *Satō* [28], *Yano* [32], [36]). *For a vector field v^h in a compact orientable Riemannian manifold M, we have*

$$\begin{aligned}&\int_M \left(g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i + \frac{n-2}{n}\nabla^h\nabla_i v^i\right)v_h dV \\ &\quad + \frac{1}{2}\int_M \left(\nabla^j v^i + \nabla^i v^j - \frac{2}{n}\nabla_t v^t g^{ji}\right) \\ &\quad\quad \cdot \left(\nabla_j v_i + \nabla_i v_j - \frac{2}{n}\nabla_s v^s g_{ji}\right)dV = 0\ .\end{aligned} \tag{2.1}$$

Proof. By a straightforward computation, we have

$$\begin{aligned}\nabla_i\left[\left(\nabla^i v^h + \nabla^h v^i - \frac{2}{n}\nabla_t v^t g^{ih}\right)v_h\right] &= \left(g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i + \frac{n-2}{n}\nabla^h\nabla_i v^i\right)v_h \\ &\quad + \frac{1}{2}\left(\nabla^j v^i + \nabla^i v^j - \frac{2}{n}\nabla_t v^t g^{ji}\right)\left(\nabla_j v_i + \nabla_i v_j - \frac{2}{n}\nabla_s v^s g_{ji}\right),\end{aligned}$$

and consequently, integrating over M we have (2.1).

Remark. If a vector field v^h defines an infinitesimal conformal transformation, then we have (1.2), i.e.,

$$\nabla_j v_i + \nabla_i v_j - \frac{2}{n}\nabla_t v^t g_{ji} = 0 . \tag{2.2}$$

From this, we can deduce

$$g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i + \frac{n-2}{n}\nabla^h\nabla_i v^i = 0 . \tag{2.3}$$

Formula (2.1) shows that this is not only necessary but also sufficient in order that the vector field v^h define an infinitesimal conformal transformation in a compact orientable Riemannian manifold.

Lemma 2 (*Yano* [33]). *For a function ρ in a compact orientable Riemannian manifold M, we have*

$$\begin{aligned}&\int_M \left(g^{ji}\nabla_j\nabla_i\rho^h + K_i{}^h\rho^i + \frac{n-2}{n}\nabla^h\Delta\rho\right)\rho_h dV \\ &\quad + 2\int_M \left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)\left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)dV = 0 ,\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\int_M \left[(g^{ji}\nabla_j\nabla_i\rho^h + K_i{}^h\rho^i)\rho_h - \frac{n-2}{n}(\Delta\rho)^2\right]dV \\ &\quad + 2\int_M \left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)\left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)dV = 0 ,\end{aligned} \tag{2.5}$$

where $\rho_i = \nabla_i\rho$, $\rho^h = \rho_i g^{ih}$ and $\Delta\rho = g^{ji}\nabla_j\nabla_i\rho$.

Proof. Putting $v^h = \rho^h$ in (2.1) and using $\nabla^j\rho^i = \nabla^i\rho^j$, we obtain (2.4). (2.5) follows from (2.4) because of

$$\int_M (\nabla^h\Delta\rho)\rho_h dV = -\int_M (\Delta\rho)^2 dV . \tag{2.6}$$

Lemma 3 (*Yano* [33]). *For a function ρ in a Riemannian manifold M, we have*

$$\nabla^h\Delta\rho = g^{ji}\nabla_j\nabla_i\rho^h - K_i{}^h\rho^i , \tag{2.7}$$

that is,

$$g^{ji}\nabla_j\nabla_i\rho^h = \nabla^h\Delta\rho + K_i{}^h\rho^i . \tag{2.8}$$

Proof. We have

$$\begin{aligned}\nabla_h \Delta\rho &= \nabla_h(g^{ji}\nabla_j\rho_i) = g^{ji}\nabla_h\nabla_j\rho_i \\ &= g^{ji}(\nabla_j\nabla_h\rho_i - K_{hji}{}^t\rho_t) = g^{ji}\nabla_j\nabla_i\rho_h - K_h{}^t\rho_t\ ,\end{aligned}$$

from which (2.7) follows.

Lemma 4. *For a function ρ in a compact orientable Riemannian manifold M, we have*

$$\begin{aligned}&\int_M \left(K_{ji}\rho^j\rho^i + \frac{n-1}{n}\rho^h\nabla_h\Delta\rho\right)dV \\ &\quad + \int_M \left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)\left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)dV = 0\ ,\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\int_M \left[K_{ji}\rho^j\rho^i - \frac{n-1}{n}(\Delta\rho)^2\right]dV \\ &\quad + \int_M \left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)\left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)dV = 0\ .\end{aligned} \tag{2.10}$$

Proof. Substituting (2.8) in (2.4) we have (2.9), and substituting (2.8) in (2.5) we have (2.10).

Lemma 5 (*Yano* [31]). *For an infinitesimal conformal transformation v^h in a Riemannian manifold, we have*

$$\mathcal{L}_v K_{kji}{}^h = -\delta_k^h\nabla_j\rho_i + \delta_j^h\nabla_k\rho_i - (\nabla_k\rho^h)g_{ji} + (\nabla_j\rho^h)g_{ki}\ , \tag{2.11}$$

$$\mathcal{L}_v K_{ji} = -(n-2)\nabla_j\rho_i - \Delta\rho g_{ji}\ , \tag{2.12}$$

$$\mathcal{L}_v K = -2(n-1)\Delta\rho - 2K\rho\ . \tag{2.13}$$

Proof. We can prove these using (1.2) and the following formulas for Lie derivatives:

$$\mathcal{L}_v\{{}_j{}^h{}_i\} = \delta_j^h\rho_i + \delta_i^h\rho_j - g_{ji}\rho^h\ , \tag{2.14}$$

$$\mathcal{L}_v K_{kji}{}^h = \nabla_k\mathcal{L}_v\{{}_j{}^h{}_i\} - \nabla_j\mathcal{L}_v\{{}_k{}^h{}_i\}\ , \tag{2.15}$$

$\{{}_j{}^h{}_i\}$ being Christoffel symbols formed with g_{ji}.

Lemme 6. *For an infinitesimal conformal transformation v^h in a Riemannian manifold M satisfying* (1.2), *we have*

$$\mathcal{L}_v G_{ji} = -(n-2)\left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)\ , \tag{2.16}$$

$$\begin{aligned}\mathcal{L}_v Z_{kji}{}^h &= -\delta_k^h\nabla_j\rho_i + \delta_j^h\nabla_k\rho_i - (\nabla_k\rho^h)g_{ji} + (\nabla_j\rho^h)g_{ki} \\ &\quad + \frac{2}{n}\Delta\rho(\delta_k^h g_{ji} - \delta_j^h g_{ki})\ ,\end{aligned} \tag{2.17}$$

where G_{ji} *and* $Z_{kji}{}^h$ *are given by* (1.4) *and* (1.7) respectively.

Proof. (2.16) follows from (2.12) and (2.13), and (2.17) follows from (2.11) and (2.13).

Lemma 7. *If a compact orientable Riemannian manifold M of dimension* $n > 2$ *admits an infinitesimal conformal transformation* v^h *satisfying* (1.2), *then*

$$\Delta\rho = -\frac{1}{n-1}K\rho - \frac{1}{2(n-1)}\mathcal{L}_v K\,, \tag{2.18}$$

$$\int_M K\rho dV = 0\,, \tag{2.19}$$

$$\int_M \mathcal{L}_v K dV = 0\,. \tag{2.20}$$

Proof. (2.18) follows from (2.13). Using (2.18),

$$\int_M \Delta f dV = 0\,, \qquad (f\text{: a scalar field on } M) \tag{2.21}$$

for $f = \rho$,

$$\mathcal{L}_v K = v^i \nabla_i K\,, \tag{2.22}$$

$$\nabla_i v^i = n\rho \tag{2.23}$$

and $\nabla_i(v^i K) = K\nabla_i v^i + v^i\nabla_i K$, and applying the well-known Green's formula we readily obtain (2.19), which together with (2.18) and (2.21) for $f = \rho$ implies (2.20). It should be remarked that (2.20) shows that if $\mathcal{L}_v K =$ const. then $\mathcal{L}_v K = 0$.

Lemma 8. *If a compact orientable Riemannian manifold M of dimension* $n \geq 2$ *admits an infinitesimal conformal transformation* v^h *satisfying* (1.2), *then*

$$\int_M g_{ji}\rho^j\rho^i dV = \frac{1}{n-1}\int_M K\rho^2 dV + \frac{1}{2(n-1)}\int_M (\mathcal{L}_v K)\rho dV\,. \tag{2.24}$$

Proof. (2.24) follows from integration over M of

$$\frac{1}{2}\Delta(\rho^2) = (\Delta\rho)\rho + g_{ji}\rho^j\rho^i\,, \tag{2.25}$$

and use of (2.18) and (2.21) for $f = \rho^2$.

Remark. If a compact orientable Riemannian manifold with $K =$ const. admits an infinitesimal nonhomothetic conformal transformation v^h satisfying

(1.2), then (2.24) implies that $K \geq 0$, and therefore that $K = 0$ (Kurita [20]) since otherwise $\rho_i = 0$ which means that v^h is homothetic.

Lemma 9. *If a compact orientable Riemannian manifold M admits an infinitesimal conformal transformation v^h satisfying* (1.2), *then*

$$\int_M K_{ji}v^j\rho^i dV + (n-1)\int_M g_{ji}\rho^j\rho^i dV = 0 . \tag{2.26}$$

Proof. Using (2.22), (2.2), (2.23), (2.13), (2.25) and

$$\nabla^j K_{ji} = \tfrac{1}{2}\nabla_i K , \tag{2.27}$$

by direct covariant differentiation we easily obtain

$$\nabla^j(K_{ji}v^i\rho) = -\tfrac{1}{2}(n-1)\Delta(\rho^2) + (n-1)g_{ji}\rho^j\rho^i + K_{ji}v^j\rho^i .$$

Thus integrating this over M, we obtain (2.26).

Lemma 10. *If a compact orientable Riemannian manifold M of dimension $n > 2$ admits an infinitesimal conformal transformation v^h satisfying* (1.2), *then*

$$\begin{aligned} &\int_M K_{ji}\rho^j\rho^i dV - \frac{1}{4n(n-1)}\int_M (2K\rho + \mathscr{L}_v K)^2 dV \\ &= \frac{1}{n-2}\int_M \left[2\rho^2 G_{ji}G^{ji} + \frac{1}{2}\rho\mathscr{L}_v(G_{ji}G^{ji})\right]dV \\ &+ \frac{1}{2}\int_M \left\{K\rho_i\rho^i - \frac{1}{2n(n-1)}[2nK^2\rho^2 + (n+2)K\rho\mathscr{L}_v K + (\mathscr{L}_v K)^2]\right\}dV . \end{aligned} \tag{2.28}$$

Proof. Substituting (2.16) in

$$\mathscr{L}_v(G_{ji}G^{ji}) = 2(\mathscr{L}_v G_{ji})G^{ji} - 4\rho G_{ji}G^{ji} ,$$

and using $g_{ji}G^{ji} = 0$ and (1.4) we obtain

$$K_{ji}\nabla^j\rho^i = -\frac{1}{n-2}\left[2\rho G_{ji}G^{ji} + \frac{1}{2}\mathscr{L}_v(G_{ji}G^{ji})\right] + \frac{1}{n}K\Delta\rho . \tag{2.29}$$

On the other hand, direct covariant differentiation gives

$$\nabla^j(K_{ji}\rho\rho^i) = \tfrac{1}{2}(\nabla_i K)\rho\rho^i + K_{ji}\rho^j\rho^i + \rho K_{ji}\nabla^j\rho^i , \tag{2.30}$$

$$\nabla_i(K\rho\rho^i) = (\nabla_i K)\rho\rho^i + K\rho_i\rho^i + K\rho\Delta\rho , \tag{2.31}$$

where we have used (2.27) for (2.30). Eliminating $K_{ji}\nabla^j\rho^i$ and $(\nabla_i K)\rho\rho^i$ from

(2.29), (2.30) and (2.31), integrating the resulting equation over M, and using (2.13) we can easily obtain

$$\begin{aligned}(2.32)\qquad \int_M K_{ji}\rho^j\rho^i dV &= \frac{1}{n-2}\int_M \left[2\rho^2 G_{ji}G^{ji} + \frac{1}{2}\rho\mathscr{L}_v(G_{ji}G^{ji})\right]dV \\ &+ \frac{1}{2}\int_M K\rho_i\rho^i dV - \frac{n-2}{4n(n-1)}\int_M K\rho(2K\rho + \mathscr{L}_v K)dV .\end{aligned}$$

Thus substracting

$$\begin{aligned}(2.33)\qquad &\frac{1}{4n(n-1)}\int_M (2K\rho + \mathscr{L}_v K)^2 dV \\ &= \frac{1}{4n(n-1)}\int_M [4K^2\rho^2 + 4K\rho\mathscr{L}_v K + (\mathscr{L}_v K)^2]dV\end{aligned}$$

from (2.32), we reach (2.28).

Lemma 11. *If a compact orientable Riemannian manifold M of dimension $n \geq 2$ admits an infinitesimal conformal transformation v^h satisfying* (1.2), *then*

$$\begin{aligned}(2.34)\qquad &\int_M K_{ji}\rho^j\rho^i dV - \frac{1}{4n(n-1)}\int_M (2K\rho + \mathscr{L}_v K)^2 dV \\ &= \frac{1}{2}\int_M \left[\rho^2 Z_{kjih}Z^{kjih} + \frac{1}{4}\rho\mathscr{L}_v(Z_{kjih}Z^{kjih})\right]dV \\ &+ \frac{1}{2}\int_M \Big\{K\rho_i\rho^i - \frac{1}{2n(n-1)}[2nK^2\rho^2 \\ &\qquad + (n+2)K\rho\mathscr{L}_v K + (\mathscr{L}_v K)^2]\Big\}dV .\end{aligned}$$

Proof. Substituting (2.17) in

$$\mathscr{L}_v(Z_{kjih}Z^{kjih}) = 2(\mathscr{L}_v Z_{kji}{}^h)Z^{kji}{}_h - 4\rho Z_{kjih}Z^{kjih} ,$$

and using (2.13), $Z_{tji}{}^t = G_{ji}$, $g_{ji}G^{ji} = 0$ we find

$$\mathscr{L}_v(Z_{kjih}Z^{kjih}) = -8G_{ji}\nabla^j\rho^i - 4\rho Z_{kjih}Z^{kjih} ,$$

or, in consequence of (1.4),

$$(2.35)\qquad K_{ji}\nabla^j\rho^i = -\frac{1}{2}\rho Z_{kjih}Z^{kjih} - \frac{1}{8}\mathscr{L}_v(Z_{kjih}Z^{kjih}) + \frac{1}{n}K\Delta\rho .$$

On the other hand, using (2.27) and direct covariant differentiation we have

$$\nabla^j(K_{ji}\rho\rho^i) = \tfrac{1}{2}(\nabla_i K)\rho\rho^i + K_{ji}\rho^j\rho^i + \rho K_{ji}\nabla^j\rho^i . \tag{2.36}$$

Eliminating $K_{ji}\nabla^j\rho^i$ and $(\nabla_i K)\rho\rho^i$ from (2.35), (2.36) and (2.31), integrating the resulting equation over M, and using (2.13) we can easily obtain

$$\begin{aligned}\int_M K_{ji}\rho^j\rho^i dV &= \frac{1}{2}\int_M \left[\rho^2 Z_{kjih}Z^{kjih} + \frac{1}{4}\rho\mathscr{L}_v(Z_{kjih}Z^{kjih})\right]dV \\ &+ \frac{1}{2}\int_M K\rho_i\rho^i dV - \frac{n-2}{4n(n-1)}\int_M K\rho(2K\rho + \mathscr{L}_v K)dV .\end{aligned} \tag{2.37}$$

Thus substracting (2.33) from (2.37) we reach (2.34).

3. Propositions

Proposition 1. *If a compact Riemannian manifold M of dimension $n \geq 2$ admits a nonconstant function ρ, then*

$$\frac{1}{n}(\Delta\rho)^2 \leq (\nabla^j\rho^i)(\nabla_j\rho_i) , \tag{3.1}$$

equality holding if and only if M is conformal to a sphere.

Proof. (3.1) is equivalent to

$$\left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)\left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right) \geq 0 ,$$

equality holding if and only if (1.14) holds, that is, by Theorem N, if and only if M is conformal to a sphere.

Proposition 2. *If a complete Riemannian manifold M of dimension $n > 2$ admits a complete infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2), *then*

$$\frac{1}{4n(n-1)^2}(2K\rho + \mathscr{L}_v K)^2 \leq (\nabla^j\rho^i)(\nabla_j\rho_i) , \tag{3.2}$$

equality holding if and only if M is isometric to a sphere.

Proof. (3.2) follows from (2.13) and (3.1) immediately, and the equality holds if and only if (1.14) does, that is, by Theorem S, if and only if M is isometric to a sphere.

Remark. If $\mathscr{L}_v K = 0$, then (3.2) becomes (1.8), and consequently Proposition 2 generalizes Theorem H.

Proposition 3. *If a compact Riemannian manifold M of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2) *such that*

$$\nabla_j\nabla_i F = \tfrac{1}{2}(2K\rho + \mathscr{L}_v K)g_{ji} \tag{3.3}$$

for a certain function F on M, then M is isometric to a sphere.

Proof. From (3.3) and (2.13) we find

$$\nabla_j \nabla_i F = -(n-1)\Delta\rho g_{ji} , \tag{3.4}$$

which implies $\Delta[F + n(n-1)\rho] = 0$, and consequently $F + n(n-1)\rho =$ const., from which it follows that

$$\nabla_j \nabla_i F + n(n-1)\nabla_j \nabla_i \rho = 0 . \tag{3.5}$$

Comparison of (3.5) with (3.4) gives (1.14). Thus, by Theorem S, M is isometric to a sphere.

Proposition 3 generalizes Theorem R (1).

Proposition 4. *If a compact orientable Riemannian manifold M of dimension $n \geq 2$ admits a nonconstant function ρ such that*

$$K_i{}^h \rho^i + \frac{n-1}{n} \nabla^h \Delta\rho = 0 , \tag{3.6}$$

then M is conformal to a sphere.

Proof. Multiplying (3.6) by 2 and adding the resulting equation to (2.7), we obtain (2.3). Thus by the remark on Lemma 1 we see that ρ^h defines an infinitesimal conformal transformation and consequently that (1.14) holds. Hence, by Theorem N, M is conformal to a sphere.

Proposition 5. *If a compact Riemannian manifold M of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2) *and* (3.6), *then M is isometric to a sphere.*

Proof. From the proof of Proposition 4, M admits an infinitesimal nonhomothetic conformal transformation v^h satisfying (1.2) and (1.14), and consequently, by Theorem S, M is isometric to a sphere.

Remark. If $\mathscr{L}_v K = 0$, then due to (2.13) the condition (3.6) becomes the first equation of (1.19). Thus Proposition 5 generalizes Theorem R (2).

Proposition 6. *If a compact orientable Riemannian manifold M of dimension $n \geq 2$ admits a nonconstant function ρ, then*

$$\int_M K_{ji}\rho^j \rho^i dV \leq \frac{n-1}{n} \int_M (\Delta\rho)^2 dV , \tag{3.7}$$

equality holding if and only if M is conformal to a sphere.

Proof. (3.7) follows from (2.10), and the equality holds if and only if (1.14) does, that is, if and only if M is conformal to a sphere.

Corollary. *If a compact orientable Riemannian manifold M of dimension $n \geq 2$ admits a nonconstant function ρ such that*

$$\int_M \left[K_{ji}\rho^j \rho^i - \frac{n-1}{n}(\Delta\rho)^2\right] dV \geq 0 , \tag{3.8}$$

then M is conformal to a sphere.

Proposition 7. *If a compact orientable Riemannian manifold M of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2), *then*

$$\int_M K_{ji}\rho^j\rho^i dV \leq \frac{1}{4n(n-1)}\int_M (2K\rho + \mathscr{L}_v K)^2 dV , \tag{3.9}$$

equality holding if and only if M is isometric to a sphere.

Proof. This follows from (2.5), (2.13) and Theorem S.

From Proposition 7, we have

Proposition 8. *If a compact orientable Riemannian manifold M of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2) *such that*

$$\int_M [K_{ji}\rho^j\rho^i - \frac{1}{4n(n-1)}(2K\rho + \mathscr{L}_v K)^2] dV \geq 0 , \tag{3.10}$$

then M is isometric to a sphere.

If $\mathscr{L}_v K = 0$, then (3.10) becomes (1.12), and consequently Proposition 8 generalizes Theorem L. For this generalization, see also Ackler and Hsiung [1].

If moreover $K = \text{const.}$, then (1.3) follows from (2.24) and (1.12). Thus Proposition 8 generalizes Theorem F.

Proposition 9. *If a compact orientable Riemannian manifold M of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation v^h satsfying* (1.2) *and* (1.15) *with a constant k satisfying*

$$(2K\rho + \mathscr{L}_v K)^2 \leq 4n^2k^2\rho^2 , \tag{3.11}$$

then M is isometric to a sphere.

Proof. Substituting (1.15) in (2.26), eliminating $\int_M \rho_i v^i dV$ from the resulting equation and the equation obtained by integrating $\nabla_i(\rho v^i) = \rho\nabla_i v^i + \rho_i v^i$ over M, and using (2.23) we readily obtain

$$nk\int_M \rho^2 dV = (n-1)\int_M g_{ji}\rho^j\rho^i dV . \tag{3.12}$$

On the other hand, from (1.15), (3.11) and (3.12) it follows that

$$\begin{aligned}\int_M K_{ji}\rho^j\rho^i dV = k\int_M g_{ji}\rho^j\rho^i dV &= \frac{n}{n-1}k^2\int_M \rho^2 dV \\ &\geq \frac{1}{4n(n-1)}\int_M (2K\rho + \mathscr{L}_v K)^2 dV .\end{aligned}$$

Thus, by Proposition 8, M is isometric to a sphere.

If $\mathscr{L}_v K = 0$, then (3.11) becomes (1.16), and consequently Proposition 9 generalizes Theorem P.

Proposition 10. *If a complete Riemannian manifold M of dimension $n > 2$ admits a complete infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2) *and* (1.20), *then M is isometric to a sphere.*

Proof. From (2.12) and (1.20) we have

$$\nabla_j \rho_i = -\frac{1}{n-2}(\alpha + \Delta\rho) g_{ji} ,$$

and consequently, by Theorem S, M is isometric to a sphere.

Proposition 10 generalizes Theorem R (3).

Proposition 11. *If a compact orientable Riemanian manifold M of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2), (1.5) *and*

$$\begin{aligned} &\int_M k\rho_i \rho^i dV \\ (3.13) \qquad &\geq \frac{1}{2n(n-1)} \int_M [2nK^2\rho^2 + (n+2)K\rho\mathscr{L}_v K + (\mathscr{L}_v K)^2] dV , \end{aligned}$$

then M is isometric to a sphere.

Proof. Under these assumptions, (2.28) implies (3.10), and consequently Proposition 11 follows from Proposition 8.

If $\mathscr{L}_v K = 0$, then (3.13) reduces to (1.13), and consequently Proposition 11 generalizes the first part of Theorem M.

Proposition 12. *If a compact orientable Riemannian manifold M of dimension $n > 2$ admits an infinitesimal nonhomothetic conformal transformation v^h satisfying* (1.2), (1.6) *and* (3.13), *then M is isometric to a sphere.*

Proof. Under these assumptions, (2.34) implies (3.10), and consequently Proposition 12 follows from Proposition 8.

Proposition 12 generalizes the second part of Theorem M.

Bibliography

[1] L. L. Ackler & C. C. Hsiung, *Isometry of Riemannian manifolds to spheres,* Ann. Mat. Pura Appl. **99** (1974) 53–64.

[2] C. Barbance, *Transformations conformes d'une variété riemannienne compacte,* C. R. Acad. Sci. Paris **260** (1965) 1547–1549.

[3] R. L. Bishop & S. I. Goldberg, *A characterization of the Euclidean sphere,* Bull. Amer. Math. Soc. **72** (1966) 122–124.

[4] S. I. Goldberg, *Manifolds admitting a one-parameter group of conformal transformations,* Michigan Math. J. **15** (1968) 339–344.

[5] S. I. Goldberg & S. Kobayashi, *The conformal transformation group of a compact Riemannian manifold,* Proc. Nat. Acad. Sci. U.S.A. **48** (1962) 25–26.

[6] ——, *The conformal transformation group of a compact Riemannian manifold*, Amer. J. Math. **84** (1962) 170–174.
[7] ——, *The conformal transformation group of a compact homogeneous Riemannian manifold*, Bull. Amer. Math. Soc. **68** (1962) 378–381.
[8] S. I. Goldberg & K. Yano, *Manifolds admitting a non-homothetic conformal transformation*, Duke Math. J. **37** (1970) 655–670.
[9] H. Hiramatu, *On integral inequalities in Riemannian manifolds admitting a one-parameter conformal transformation group*, to appear in Kōdai Math. Sem. Rep.
[10] ——, *On Riemannian manifolds admitting a one-parameter conformal transformation group*, Tensor, **28** (1974) 19–24.
[11] C. C. Hsiung, *On the group of conformal transformations of a compact Riemannian manifold*, Proc. Nat. Acad. Sci. U.S.A. **56** (1965) 1509–1513.
[12] ——, *On the group of conformal transformations of a compact Riemannian manifold*. II, Duke Math. J. **34** (1967) 337–341.
[13] ——, *On the group of conformal transformations of a compact Riemannian manifold*. III, J. Differential Geometry **2** (1968) 185–190.
[14] C. C. Hsiung & J. D. Liu, *The group of conformal transformations of a compact Riemannian manifold*, Math. Z. **105** (1968) 307–312.
[15] C. C. Hsiung & L. R. Mugridge, *Conformal changes of metrics on a Riemannian manifold*, Math. Z. **119** (1971) 179–187.
[16] C. C. Hsiung & L. W. Stern, *Conformality and isometry of Riemannian manifolds to spheres*, Bull. Amer. Math. Soc. **163** (1970) 1253–1256.
[17] ——, *Conformality and isometry of Riemannian manifolds to spheres*, Trans. Amer. Math. Soc. **163** (1972) 65–73.
[18] S. Ishihara, *On infinitesimal concircular transformations*, Kōdai Math. Sem. Rep. **12** (1960) 45–56.
[19] S. Ishihara & Y. Tashiro, *On Riemannian manifolds admitting a concircular transformation*, Math. J. Okayama Univ. **9** (1959) 19–47.
[20] M. Kurita, *A note on conformal mappings of certain Riemannian manifolds*, Nagoya Math. J. **21** (1962) 111–114.
[21] A. Lichnerowicz, *Transformations infinitésimales conformes de certaines variétés riemanniennes compacte*, C. R. Acad. Sci. Paris **241** (1955) 726–729.
[22] ——, *Sur les transformations conformes d'une variété riemannienne compacte*, C. R. Acad. Sci. Paris **259** (1964) 697–700.
[23] T. Nagano, *The conformal transformations on a space with parallel Ricci curvature*, J. Math. Soc. Japan **11** (1959) 10–14.
[24] M. Obata, *Conformal transformations of compact Riemannian manifolds*, Illinois J. Math. **6** (1962) 292–295.
[25] ——, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** (1962) 333–340.
[26] ——, *Riemannian manifolds admitting a solution of a certain system of differential equations*, Proc. United States-Japan seminar in Differential Geometry, Kyoto, Japan, 1965, Nippon Hyoronsha, Tokyo, 1966, 101–114.
[27] ——, *Quelques inégalités intégrales sur une variété riemannienne compacte*, C. R. Acad. Sci. Paris **264** (1967) 121–125.
[28] I. Satō, *On conformal Killing tensor fields*, Bull. Yamagata Univ. **3** (1956) 175–180.
[29] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117** (1965) 251–275.
[30] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960) 21–37.
[31] K. Yano, *The theory of Lie derivatives and its applications*, North-Holland, Amusterdam, 1957.
[32] ——, *Some integral formulas and their applications*, Michigan J. Math. **5** (1958) 63–73.
[33] ——, *On Riemannian manifolds with constant scalar curvature admitting a conformal transformation group*, Proc. Nat. Acad. Sci. U.S.A. **55** (1966) 472–476.
[34] ——, *Riemannian manifolds admitting a conformal transformation group*, Proc.

Nat. Acad. Sci. U.S.A. **62** (1969) 314–319.
[35] ——, *On Riemannian manifolds admitting an infinitesimal conformal transformation,* Math. Z. **113** (1970) 205–214.
[36] ——, *Integral formulas in Riemannian geometry,* Marcel Dekker, New York, 1970.
[37] ——, *Conformal transformations in Riemannian manifolds,* Differentialgeometrie im Grossen, Berichte Math. Forschungsinst., Oberwolfach, 4, 1971, 339–351.
[38] K. Yano & T. Nagano, *Einstein spaces admitting a one-parameter group of conformal transformations,* Ann. of Math. **69** (1959) 451–461.
[39] K. Yano & M. Obata, *Sur le groupe de transformations conformes d'une variété de Riemann dont le scalaire de courbure est constant,* C. R. Acad. Sci. Paris **260** (1965) 2698–2700.
[40] ——, *Conformal changes of Riemannian metrics,* J. Differential Geometry **4** (1970) 53–72.
[41] K. Yano & S. Sawaki, *On Riemannian manifolds admitting a conformal transformation group,* J. Differential Geometry **2** (1968) 161–184.
[42] ——, *Riemannian manifolds admitting an infinitesimal conformal transformation,* Kōdai Math. Sem. Rep. **22** (1970) 272–300.

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